$AF = AC$, and from the equality of the right triangles BaE and BaC it follows that $BE = BC$.

THEOREM III. *The radius r of the circle inscribed in the triangle ABC is equal to* $\frac{1}{2}EF = ab$.

Proof. Since $CO = OF$ (ΔCOF is isosceles) and $CO = OE$ (ΔCOE is isosceles), $OE = OF$; and consequently EFO is an isosceles triangle.

Moreover, $\underline{|FOE} = 2\underline{|FCE} = 90°$. Whence, $OP \equiv r = \frac{1}{2}EF = ab$.

THEOREM IV. *The radius,* $r = cd/\sqrt{2}$, *i. e., the radius is equal to the side of a square whose diagonal is cd.*

Proof. From the similarity of the triangles aOb and dOc, we have

$$\frac{ab}{cd} = \frac{Oa}{Oc} = \frac{Ob}{Od} = \sin 45° = \frac{1}{\sqrt{2}}.$$

Hence (Theorem III), $r = ab = \dfrac{cd}{\sqrt{2}}$.

THEOREM V. *The line of centers of the circles inscribed in the triangles ADC and BDC is perpendicular to the bisector of the right angle C, i. e.,* $cd \perp CG$.

Proof. Since in the triangle cdC two altitudes cb and da pass through the point O, $CG \perp cd$.

THEOREM VI. *The centers of the circles inscribed in the triangles ADC and DBC, i. e., the points c and d are equidistant from the point P and their distances are equal to* $OP = r$.

Proof. From the equality of triangles OFd and OcE it follows that $OD = Ec$, and hence follows the equality of OPd and PEc and consequently $cP = Pd$. Moreover, $\underline{|OdF} = 180° - (\alpha + \beta) = 180° - 45° = 135°$. Hence, if a circle be drawn with P as the center and $OP = PF = r$ as the radius $\underline{|OdF}$ will be an inscribed angle and $Pd = cP = r$.

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY R. C. ARCHIBALD, Brown University, Providence, R. I.

ADDRESS.[1]

By HERBERT E. HAWKES, Columbia University.

In the remarks which I have to make this morning, I do not propose to go into details regarding the history of Mathematics Clubs or to discuss the function of such clubs under ideal conditions. I wish rather to tell briefly of my own experience with such clubs which may possibly serve as a practical help to those who have conducted or are about to organize such organizations in their own institutions.

[1] Delivered at the second summer meeting of the Mathematical Association of America, at Cleveland, Ohio, September 6, 1917.

At the very start let me say that the kind of club I have in mind is not the kind which is of interest merely to the two or three ablest men in college. The exceptional man who even in his undergraduate days shows promise of unusual scientific enthusiasm and ability, the kind of man who appears once in several years in the small college, and of whom there is only one in a class in the large college, must be taken in charge by his instructor in any case, and should have careful personal direction so that if he is the kind of man who should be advised to study mathematics professionally, he may be brought in contact with the real thing in mathematical investigation as early as practicable. Such a man ought to be a leader in the club, but if the club is geared for this man, it will be a flat failure. The pace will certainly be too hot for his less gifted fellows, and the result will be a mutual admiration society of about three somewhat unworldly spirits.

The kind of club I *am* talking about is one in which the able freshman can find some interest, and every student who is interested in interesting things and who has taken up the study of analytics can take an active part.

What is the use and the function of such a club as I have indicated? It seems to me that there are three kinds of purpose that such a club may serve, if it is successful. It surely benefits those students who present papers or solutions, it ought to help those who attend, and it should be a good thing for the college as a whole.

Just a word on this last point, before taking up the first two. One of the most unfortunate traditions among the undergraduates of our colleges is the feeling that they are serving their college only when they are playing baseball, or running the football team. Their loyalty is splendid and genuine, and such a precious enthusiasm should not be allowed to cool, even if it is sometimes observed to run in a direction of secondary importance. It is almost impossible for the undergraduate to realize that in the long run the greatest honor to a college is a competent and well-trained graduate, and the most far-sighted loyalty to his alma mater would lead him to prepare himself to be such a graduate. To be an officer of the mathematics club is an honor, and those who work for the welfare of the club are recognized as working for the college. And since its activities have to do with the intellectual life, it should appeal to the most genuine spirit of loyalty to the institution.

Let me emphasize, however, that unless the activities of the club are kept simple, so that it may have the reputation in college of being well within the domain of rationality of a good many students, this feature of the club will be a failure. For a small body of pale people talking about existence theorems is not a spectacle over which college students as a whole will enthuse. I am certain, however, that this idea of organizing certain extra-curricula activities which have contact with things of the mind helps to divert some of the energy of the students from an overwhelming development of athletics. I do not intend to express any lack of appreciation for athletics. Young men must have them. But few will argue that they are not carried to an extreme in many of our colleges, and that a counter-irritant should be sought.

The benefit of the club to the student who attends, but who has no stated part is fairly obvious. It depends, however, on the way the meetings are conducted, whether the auditor gets much out of it or not. In my experience it has been absolutely necessary to conduct the meetings in the most informal manner possible. The speaker expects to be interrupted, and to explain in greater detail questions of notation or demonstration or implication which anyone requests. Often the more carefully and clearly a paper is presented, the greater curiosity it awakens, and the oftener the speaker is asked to make more detailed explanation.

The fundamental fact is that students cannot be expected to go many times to a mathematics club because they feel that they ought to. They will only go because they want to, and nothing but a good time intellectually and socially will hold them. I do not mean by this that there should be meager food for thought, and much entertainment. Quite the contrary. The intellectual nourishment cannot be too good or too well prepared. But it must be adapted for their powers of digestion, and must be served with good fellowship and simplicity.

The benefit of the club to those who give papers is naturally greater than to any one else, and on them depends the success of the club to a large extent. Our responsibility as teachers of college mathematics is nearly as great in the direction of training our students in the difficult art of self expression as in the technique of mathematics. The regular class room instruction, especially in elementary work, consists to a large extent, for many of us, in patiently requiring our students to express the truth that is in them in the form of an English sentence which has a subject, a predicate and the proper qualifying clauses. Now the club affords a splendid opportunity to give a few good students, men who are worth taking pains with, excellent practice and training in the exposition of a specific topic in mathematics. I would remark parenthetically that this means an appreciable amount of painstaking and tactful work on the part of some one on the teaching staff. The students must feel that they do the whole thing, but they must be far from correct. I have rarely found the student who had really adequate terminal facilities. They always prepare papers from twice to four times as long as they should be, and find the greatest difficulty in condensing to advantage. Even if the student sees his problem in the proper perspective, it requires almost superhuman efforts to get him to express it clearly and concisely. But I am forced to say that assisting a student to get an effective method of exposition without doing the work for him is, in my opinion, scientific education of the most important and useful kind, and any means that enables one to render such service is an important feature of the college.

The second great benefit to the speaker in the mathematics club is the experience which he gets in examining the literature of his topic, and going outside his text books for his material.

The third, and the most important of all is the consciousness that comes with independent search for truth. The truth sought is usually not new to

science, but it is new to the student, and has all of the freshness of spring to him. For college students, what might be called "searchwork," that is, topics dealing with *important* questions which contain enough meat to interest any right thinking person, is much more profitable than the discussion of some topic which he has worked out for himself but which one cannot find in the books, or memoirs. In fact, I think that it is a pity for an able student to get the impression that the discovery of some little point that no one else has happened to mention, is mathematical scholarship and a substitute for a broad and wide knowledge of the subject.

My procedure with a student who has signified his desire to prepare something for the club is somewhat as follows. After finding a topic which is within his range, and in which he is likely to be interested, I give him one or two references which he is to read, and to report upon. Then I often tell him of other books in which he had better look, and urge him to go over all the literature he can find which seems to promise well. By the time this is done he can make an outline of what he proposes to say, specifying what he wishes to prove, and what is to be given without proof. The student always wants to prove almost everything, and cannot understand how a paper can omit proofs, and still be a good one, except when he is listening to it. Finally a few days before the meeting I try to go over with him in more detail the plan of his paper, and often the details of exposition. Of course he never speaks from notes, but puts diagrams on the board before the meeting. I have found that if a paper is planned which the student thinks will take twenty-five minutes, it actually takes forty-five if he is not interrupted badly. If he is interrupted, he never gets through, but has to summarize the last part. I make great efforts, however, to keep the papers down to twenty-five minutes of talk.

Undoubtedly the kind of organization which would be best adapted to students of one institution might not be suited to another. I have had experience with only two types of club at Columbia College. At first there was no organization at all. Meetings were held every two weeks by those who were interested to come, but without undergraduate officers, no dues, and no machinery at all. I always presided, in the most informal manner, and it was a purely spontaneous expression of interest in mathematics on the part of those who attended. After a couple of years of this sort of procedure, the students suggested that they have a regular club with officers, dues of twenty-five cents per year to cover the exp nse of sending cards to members, and a written constitution.

So far as the essential features of the club are concerned, I cannot see that it makes the slightest difference what kind of organization is in force. The fact from which one cannot get away, is that a mathematics club will not run itself, and that it requires a great deal of careful work to get speakers who can attract a good house. If the organization is such that some member of the staff can really look after this feature of the work and at the same time keep in the background so that the students seem to be in charge, it does not make much difference whether there are many officers or few, or whether there is a constitution or not.

I am inclined to think, however, that the club is of more service to the college community if there are undergraduate officers, and definite organization. If the professors can attend not as professors but as fellow creatures of the students there is no objection. But the moment that the boys feel that another disguised recitation is in progress, and that their professors are watching their accomplishment, the club is no longer a club, but a parade ground.

The most important and at the same time the most difficult feature of the club to manage, is the program. Especially during the first part of the year, papers must be provided which will attract the students and hold their interest. I suppose that the mathematical topics which naturally attract the student who is not far advanced, are those which he thinks have some mystic significance. Something about higher dimensions, or infinity, or paradoxes. Personally I think that the greatest service which a club can perform in regard to these topics is to denude them of their mystic qualities, and to bring the ideas involved squarely and fairly into relations with sound mathematics. But students cannot perform this service, for they have not the perspective, and so far as my experience goes, make a bad matter of any topic relating to these subjects. Consequently, it usually seems wisest for an older person to talk about the fourth dimension and orders of infinity, rather than a student.

One might think papers of an historical nature admirably adapted for presentation at the club. But to my great surprise, I have had very little luck with such papers. It may be my own fault, but I have found that the students do not have a sufficiently broad and scholarly grasp of the subjects whose history they are presenting to make topics of this kind rewarding. Here again, a member of the staff does much better.

A distinguished educator once complained to me that the teachers of mathematics and physics missed a great opportunity in not humanizing the presentation of their subjects. He said that a boy might study physics two years and not know whether Boyle was an Irishman or a very painful swelling on one's person. I have made a serious effort to get students to prepare papers on the "heroes of mathematics." They like the idea immensely but when it comes to preparing a paper showing just where Napier, Newton, Archimedes or Descartes stood in relation to what preceded and what followed, and what specific problem stimulated them to carry forward their epoch making work, I have sometimes been disappointed, owing again to the lack of perspective. But on the whole such papers have been much more successful than those which attempt to treat the development of an idea or set of ideas.

At least one excellent meeting a year can be provided by asking the students to bring paradoxes or puzzles of a mathematical nature. These may be presented, and solutions requested from the audience.

I have found no difficulty in finding enough interesting topics which involve little or no calculus. Such a procedure does not seem to result in the loss of interest on the part of the more mature students, for they feel the need of a more complete perspective over their elementary mathematics, and as a matter of

fact, a paper which involves a good deal of the calculus cannot be followed in detail by as arge body of students as one likes to have at the club. And, unlike the members of more pretentious mathematical societies, students do not enjoy sitting through a long paper, no part of which they understand with the possible exception of the title. And *they* will not do it.

I have mentioned several types of topics which students find some difficulty in presenting effectively. It is much easier to do that than it is to generalize regarding the kind of paper which is successful. For here, as in most questions involving education, it all comes back to the man behind the guns. If the student himself is thoroughly interested in a topic he can do wonders with it. If not, he can spoil the best subject in the world.

On the whole the students do best on topics which are very definite, and in which there is a clear relation between the new idea which they are presenting and some thing with which they are familiar. For example, I recall a very excellent paper on the exponential and logarithmic function of complex numbers, in which the well-known ideas of many-valuedness, and periodicity as found in trigonometric functions were taken for points of departure, and these same features of the functions under discussion were exposed. Also the paradoxes which arise when one takes the wrong branch of a many-valued function affords instruction and amusement.

Imaginary lines in the plane are also interesting. Any student of analytics can prove that the distance between two points on such lines is zero, and a number of other curious features. Here the hope of success of the paper lies in connecting the new ideas with the similar ones regarding real lines.

Another kind of topic which is likely to be successful is the graphical study of processes which are more or less familiar analytically. Such subjects as the graphical solution of equation, the study of constructions which are possible with the use of the compasses and a fixed parabola, construction with a two-sided straight edge are all excellent.

Since these remarks are simply intended to start a discussion in which it is hoped that many will take part, I do not propose to follow the example of the students in talking twice as long as I should. I must, however, express my conviction that a mathematics club in a college is a powerful stimulus on the side of sound scholarship, and to my certain knowledge able men are inducted through it into the serious study of mathematics, and stimulated in their ambition not only to know the subject so far as their abilities permit but to add to it as well. I know of no activity in which a college professor of mathematics can more profitably spend some effort, than in directing a club. For not only is he helping the cause of scholarship, but he is making a subject which has the reputation of being somewhat austere, an object of living and spontaneous interest on the part of a good body of students, and giving it a place in the college community.

CLUB ACTIVITIES.

The Mathematics Club of Albion College, Albion, Mich.

This club was founded in January, 1911, "to create a greater interest in present-day mathematics and allied subjects, and to study best methods of teaching mathematics." Any student showing unusual ability during the first three semesters of the college course may be elected to membership in the club. All nominations are made by the head of the department, Professor Edwin R. Sleight.

Meetings last for about an hour and the usual routine is as follows: (1) roll-call, each member responding to his name by a short discussion (2 or 3 minutes as a maximum)[1] of an assigned topic [there is never any trouble with this part of the program]; (2) short talk of the evening, limited to 10 minutes; (3) topic of the evening, occupying 20 to 30 minutes; (4) general discussion of any of the topics of the evening; (5) critic's report (the critic is usually a student).

Officers are elected at the close of each semester for the one following; those elected for the first semester of 1917–18 were: President, Lucille Ball '18; vice-president, Esther Turnell '18; secretary-treasurer, Myrtle Speese '19; program committee: Professor Sleight, the vice-president, the secretary-treasurer, and Gertrude Landon '19. For the second semester the officers were: President, Alice Money '18; vice-president, Myrtle Speese '19; secretary-treasurer, Floyd Harper '20.

Programs for the past two years are indicated below [(1) roll call, (2) short talk, (3) evening topic]. The short talks for 1916–17 were historical. "The programs for a normal year are so arranged that each member of the club gives one short talk and one long talk each semester. The programs for 1917–18 are not a fair sample of the work done by the club. We are so rushed by having classes six days of the week, and having practically no vacations that we decided that the programs for the year should, for the most part, be merely reviews of articles found in periodicals."

October 3, 1916: (1) None; (2) "Euclid" by Jesse Campbell '18; (3) President's address "The place of mathematics in the world" by Hazel Miller '17.

October 10: (1) "A recent invention to which mathematics was applied;" (2) "Archimedes" by Gertrude Landon '18; (3) "The forward movement in mathematics" by Alice Money '18.

October 17: (1) "Integration" [each person as his name was called was sent to the board and asked to integrate a form]; (2) "Apollonius of Perga" by Myrtle Speese '19; (3) "Some unusual integrals and their solutions" by Ralph Huffer '18.

October 24: (1) "A practical application of a geometrical theorem;" (2) "Diophantus of Alexandria" by Hazel Miller '17; (3) "A comparison of geometry with mechanics" by Gladys Harger '17.

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[1] There were 19 members in 1917–18 and this number necessarily limited the time which each individual might occupy. Of the 19 members only 4 are young men. Before the war membership was about equally divided between men and women.

October 31: (1) None; (2) "Vieta" by Esther Turnell '18; (3) "Fourier's series and its application" by Professor Arthur F. Beal.

November 7: (1) "My mental attitude toward mathematics;" (2) "Descartes" by Alice Money '18; "The development of mathematics in the United States during the nineteenth century" by Myrtle Speese '19.

November 14: "Some interesting celestial objects" (illustrated lecture, open session) by Professor Beal.

November 28: (1) "A peculiar geometrical construction;" (2) "Gauss" by Vera Junkin '19; (3) "A geometrical representation of multiplication and division" by Mildred Chappell '19.

December 7: (1) "Mathematics in poetry" (quotations referring to mathematics); (2) "Newton" by Ralph Huffer '18; (3) "Mathematics and psychology" by Esther Turnell '18.

December 14: (1) "A coördinate system;" (2) "Leibnitz" by Gladys Harger '18; (3) "Homogeneous coördinate systems" by Gertrude Landon '19.

January 16, 1917: This program consisted of four short talks: "Right and wrong definition of a limit" by Lloyd Crippen '19; "Distributive law of multiplication geometrically proved" by Esther Turnell '18; "The teaching of factoring" by Gladys Harger '17; "Judging a teacher of mathematics" by Myrtle Speese '19.

January 23: (1) "Who's who in mathematics in America;" (2) "A simple solution of a Diophantine equation" by Lucille Ball '18; (3) "The circles of Apollonius" by Ralph Huffer '18.

February 6: (2) "Lagrange" by Myrtle Speese '19; (3) "Tests and short cuts in mathematics" by Jesse Campbell '18; (1) Application of the short cuts given in (3).

February 20: (1) "An interesting event in the life of some mathematician;" (2) "Euler" by Gladys Harger '17; (3) "Some simple applications of congruences" by Hazel Miller '17.

February 27: (1) "Mathematical current events;" (2) "Cauchy" by Edna Colwell '19; (3) "War and mathematics" by Vera Junkin '19.

March 6: (1) "Name of some subject in higher mathematics and its application;" (2) "Steiner" by Margaret Courtright '19; (3) "Different kinds of geometries, with special emphasis on descriptive" by Ian Patterson '19.

March 13: (1) "Mathematics applied to some definite problem;" (2) "Abel" by Mildred Chappell '19; (3) "The future of mathematics" by Mary Hutchins '19.

March 20: (1) "Some interesting personal experience in a mathematics classroom;" (2) "Hypatia" by Murray Fox '18; (3) "Relation of the teacher to the community" by Alice Money '18.

April 3: (1) "Some theorem used in algebra—statement and application;" (2) "Galois" by Gertrude Landon '19; (3) "Discussion of Fermat's theorem" by Ralph Huffer '18.

April 10: (1) "A brief review of some new text-book;" (3) Report by Edna

Colwell '19 who had been elected to attend the mathematics section of the State Schoolmasters' Club held at Ann Arbor.

April 17: (1) "Name of some mathematician connected with function theory, and a brief statement of what he did;" (3) "Teaching of mathematics in secondary schools" by Principal Harry R. Atkinson of Battle Creek, Michigan.

May 1: (1) "The name of some educational society, together with a brief statement of its purpose;" (2) "Sylvester" by Jesse Campbell '18; (3) "Variables and limits" by Lloyd Crippen '19.

May 8: (1) "Mathematical fallacies;" (2) "Weierstrass" by Ian Patterson '19; (3) "How to make the teaching of mathematics interesting" by Hazel Miller '17.

May 15: (1) "Some unusual function;" (2) "Lie" by Murray Fox '19; (3) "Eulerian integrals and gamma functions" by Lucille Ball '18.

May 22: (1) None; (2) "Origin of certain typical problems" by Esther Turnell '18; (3) "Poincaré and his contributions to mathematics" by Margaret Courtright '19.

May 29: Social evening and election of officers for the following semester.

October 16, 1917: President's address by Lucille Ball '18.

October 23: (1) "Mathematical current events;" (3) "Geometrical proof of formulas for sine, cosine and tangent of half angles" by Vera Junkin '19; "Algebraical developments in Ancient Greece and Babylonia" by Margaret Courtright.

October 30: (1) "Some theorem in college algebra proved;" (3) "Condition that three lines pass through the same point" by Joyce Hadaway '20; "Freshmen and freshmen algebra" by Esther Pearl '20.

November 6: (1) "Historical theorem known by the name of the discoverer;" (3) "Methods for solving irrational equations" by Esther Turnell '18; "The students viewpoint of calculus" by Alice Money '18.

November 13: (1) "A peculiar logarithmic combination;" (3) "Mathematics of warfare" by Ralph Huffer '18.

November 27: (1) "A peculiar geometrical construction;" (3) "A new proof for the law of sines" by Carla Kennedy '20; "Mathematics as a means of culture and discipline" by Elizabeth Hubert '20.

December 11: (1) "A mathematical fallacy;" (3) "Significance of mathematics" by Lucille Ball '18; "A historical account of mathematical induction" [1] by Myrtle Speese '19.

January 8, 1918: (1) "A recent text book;" (3) "Projective geometry—a historical account and some applications" by Mildred Chappell '19.

January 15: Election of officers.

February 26: (1) "The history and meaning of mathematical symbols;" (3) President's address: "Why mathematics should be studied in the high school" by Alice Money '18.

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[1] Cf. F. Cajori, (1) *Bulletin of the American Mathematical Society*, Vol. 15, pp. 407–408, 1909; (2) "Origin of the Name 'Mathematical Induction,'" in this MONTHLY, Vol. 25, pp. 197–201, 1918.

March 5: (1) "A statement of a problem whose solution is the irrational root of a cubic equation;" (2) "Some suggestions on the teaching of geometry" by Dorothy Tichenor '20; (3) "Solutions of the cubic" by Almira Priest '20.

March 19: (1) "A mathematical puzzle;" (2) "Valid aims and purposes for the study of mathematics in secondary schools" by Don Alexander '20; (3) "The function of mathematics in scientific research" by Carlton Sawyer '20.

April 2: (1) "An original mathematical limerick;" (2) "The Perry method" by Joyce Hadaway '20; (3) "A review of some old arithmetics" by Carla Kennedy '20.

April 16: (1) "Helpful hints to the teacher;" (2) "The heuristic method" by Vera Junkin '19; (3) "The planimeter" by Gertrude Landon '19.

April 30: Social evening, and election of officers for next semester.

THE MATHEMATICS CLUB OF GOUCHER COLLEGE, Baltimore, Maryland.

This club of young women was founded in November, 1913, in order "to promote a spirit of comradeship in the department, to stimulate interest in mathematics and to provide an opportunity to discuss many topics which are not included in the regular course." All students who have completed the first courses in analytic geometry and in calculus, given the first semester of the sophomore year, are eligible for membership. During 1917–18 the club had 22 members and the average attendance at the meetings was about 15. There are no officers but Professor Florence P. Lewis usually presides and acts as program committee. "At the close of each meeting we have very informal discussion and very light refreshments furnished usually by the speakers of the evening."

"Previous to 1916–17 we had no course on the history and teaching of mathematics and so devoted the club work to these subjects. We tried to cover the history in a very elementary general way in two years of club work. A secretary was appointed to make of the work a typewritten résumé, copies of which were given to each girl."

November 12, 1917: "The parallel axiom and modern work on foundations of geometry" by Professor Lewis.

November 26: "Trisection of an angle and the duplication of a cube" by A. Marie Whelan '18.

December 10: "A demonstration lesson in the teaching of geometry" by Miss Elizabeth White, teacher of mathematics, Eastern High School, Baltimore, Md.

January 21, 1918: "The triangle and its circles" [1] by Margaret Amig '19.

February 18: "Geometry of four dimensions" by Marguerite Lehr '19.

March 4: "The history of the invention of logarithms" by Teresa Cohen '12, graduate student of Johns Hopkins University.

[1] See this MONTHLY, 1918, p. 231, note 2.

April 15: "Construction of a pentagon" by Anna L. Ellery '20; "Fallacious proof that all triangles are equal" by Ethel H. Fox '19; "Orthogonal circles" by Effie M. Gray '18.
April 29: "Illustration of non-Euclidean geometry on the Poincaré sphere" by Professor Clara L. Bacon.
May 18: Club picnic at Herring Run.
May 20: "Graphic methods of presenting facts" by Mildred Grafflin '20; "A new theorem on equal circles"[1] by Ethel R. Carroll '20; "History of famous problems in algebra" by Ida R. Marshall '20.

TOPICS FOR CLUB PROGRAMS.

13. Constructions with a Double-Edged Ruler.

It is just about one hundred years since the French mathematician and engineer Jean Victor Poncelet wrote his famous *Traite des propriétés projectives des figures*[2] which entitles him to rank as one of the greatest contributors to the development of modern geometry. On pages 187–190 are given indications of the proofs that: *Every geometrical construction with ruler and straight edge is also possible*: (1) *with straight edge alone if a single circle and its center have been drawn in the plane* (a theorem often incorrectly attributed to Steiner as originator[3]); and (2) *with an angle*[4] (of wood say) *of given opening α, alone.*

Particular cases of the double-edged ruler (2) are: (3) a square, when $\alpha = 90°$; and (4) a ruler with parallel edges when $\alpha = 0°$

As a proof of (1) Steiner found it convenient to show that the means at his disposal were sufficient for solving the following problems:[5] (*a*) To draw through a given point a parallel to a given line; (*b*) to produce a given line segment its own length any given number of times or to divide it into any number of equal parts; (*c*) to draw through a given point a line perpendicular to a given line; (*d*) to draw through a given point a line which makes with a given line a given angle; (*e*) to bisect a given angle or to make an angle any number of times greater; (*f*) to draw from a given point in a given direction a segment equal to a given segment; (*g*) to determine the points of intersection of two circles whose radii and centers are given; and (*A*) to determine the points of intersection of a given line with a circle the position of whose center and the length of whose radius alone are given.

Recent proofs of (2), (3) and (4) have been, in effect, reduced to showing that these same constructions could be carried through by the special instruments in

[1] See this Monthly, 1918, p. 231, note 3.

[2] Paris, 1822; see also 2e éd., tome 1, 1865, pp. 181–184, 413–414. Poncelet tells us that this great work was the result of research made in 1813–14 when in Russian prisons "privé de toute espèce de livres et de secours, sur-tout distrait par les malheurs de ma patrie et les mien propres."

[3] Hence the expression, "Steiner constructions" employed in this connection should be "Poncelet constructions."

[4] "Fausse équerre."

[5] J. Steiner, *Die geometrischen Constructionen ausgeführt mittelst der geraden Linien und eines festen Kreises*, Leipzig, 1833, pp. 2–3, etc.

question. Such is the mode of procedure of Adler in 1890 and 1906,[1] of Enriques in 1903,[2] of Giacomini in 1907[3] and of Killing and Hovestadt in 1910.[4]

For separate discussion of constructions by means of a ruler with parallel edges reference may be given to papers and notes by Lebescond de Coatpont and De Tilly,[5] Marenghi, and Concina.[6]

Since a parallelogram may be drawn at once with a double-edged ruler note that the following famous problem was discussed by Lambert,[7] s'Gravesende,[8] Poncelet,[9] Tractenberg[10] and Child[11]: "Given a parallellogram, construct with straight edge alone a parallel to a given line."

Important practical applications, in surveying and warfare, of constructions involving a double-edged ruler are indicated in the remarkable little book of F. J. Servois, *Solutions peu connues de différens problèmes de géométrie-pratique* . . . A. Metz, An XII [1804], (especially pages 68–79), and in the very interesting work of Gohierre de Longchamps to which reference has been made already. Solutions of surveyors' problems with the square were given also by L. Mascheroni in his *Problemi per gli agrimensori con varie soluzioni*, In Pavia, MDCCXCIII.[12]

With ruler and compasses, only special problems of degree higher than the second can be solved, no difference how many of these instruments may be em-

[1] A. Adler, (1) "Über die zur Ausführung geometrischer Constructionsaufgaben zweiten Grades notwendigen Hilfsmittel," *Sitzungsberichte der mathematisch-naturwissenschaftlichen Classe der kaiserlichen Akademie der Wissenschaften zu Wien*, Band 99, Abtheilung IIa, Jahrgang 1890, Heft 8; (2) *Theorie der geometrischen Konstruktionen*, Leipzig, 1906, pp. 123–138.

[2] F. Enriques, *Vorlesungen über projektive Geometrie*, Leipzig, 1903, pp. 266–277; 2. Aufl. Leipzig, 1915, p. 254.

[3] A. Giacomini, "Über die Lösung der geometrischen Aufgaben mit dem Lineal und den lineal Instrumenten . . .," pp. 95–103 of *Fragen der Elementargeometrie* gesammelt und zusammengestellt von F. Enriques, II. Teil, Leipzig, 1907; Italian edition, Bologna, 1914, pp. 89–96.

[4] W. Killing und H. Hovestadt, *Handbuch des mathematischen Unterrichts*, Band I, Leipzig, 1910, pp. 194–199.

[5] "Sur la géométrie de la règle," *Nouvelle correspondence mathématique*, tome 3, 1877, pp. 204–208; tome 5, 1879, pp. 439–442; tome 6, 1880, pp. 34–35.

[6] C. Marenghi, "Geometria della riga a due orli paralleli," *Il bolletino di matematiche e di scienze fisiche e naturali*, Bologna, Vol. 2, 1900–1901, pp. 129–145; U. Concina "Resoluzione dei problemi fondamentali relativi al trasporto delle figure piane colla riga a due orli paralleli," *idem*, 1901, pp. 225–237.

[7] J. H. Lambert, *Freie Perspective*, 2e éd., Zürich, Vol. 2, p. 169. This solution is given also in: (1) *Mathematical Questions and Solutions from the 'Educational Times,'* Vol. 57, London, 1892, p. 88; and (2) J. W. Russell, *An Elementary Treatise on Pure Geometry*, New and revised edition, Oxford, 1905, p. 318.

[8] W. J. s'Gravesende [died 1742], *Oeuvres philosophiques*, Amsterdam, 1774, Ire partie, § 312, p. 174.

[9] *Traité des propriétés*, etc., 1822, pp. 106–107; Lambert's solution is also given here. Both of these solutions are reproduced in L. Cremona, *Elements of Projective Geometry*, translated by C. Leudesdorf, Oxford, 1885, pp. 96–97; Poncelet gave his solution of the *Traité* and two others in his *Applications d'analyse et de geométrie*, Paris, 1862, pp. 437–439 (this part of the work was written while he was in prison in 1813–14). One of these solutions is also reproduced in G. De Longchamps, *Essai sur la géométrie de la règle et de l'équerre*, Paris, 1890, pp. 232–235.

[10] M. I. Trachtenberg, *Mathematical Gazette*, 1908, Vol. 4, p. 334.

[11] J. M. Child, *Mathematical Gazette*, 1910, Vol. 5, pp. 283–284.

[12] Second edition with title: *Problemi di geometria colle dimostrazioni del Capitaus Sacchi*, Milano, 1803; third edition, 1832. French translation of the first edition: *Problêmes pour les arpenteurs avec différentes solutions*, Paris, An XI–1803; 2e éd., with title: *Problèmes de géométrie pratique pour les arpenteurs avec différentes solutions*, Paris, 1838.

ployed. In contrast to this it is interesting to note, as Adler (*l. c.*) emphasizes, that the right angle (square) is a much more powerful instrument. Indeed Plato's instrument for the solution of the problem of the duplication of a cube[1] is equivalent to the use of two right angles to solve a binomial cubic equation. So also four right angles could be used to solve a fifth degree binomial. Descartes employed a similar instrument with right angles[2] for the insertion of several geometric means between two line segments.

The fundamental ideas here introduced were adapted by Lill[3] so as to extend to the solution of polynomial equations with numerical coefficients. From this Adler readily formulated, in particular, the result: Problems of the third and fourth degree can be rigorously solved geometrically by means of several right angles.

COLLEGIATE MATHEMATICS FOR WAR SERVICE.

SEND WAR SERVICE COMMUNICATIONS TO DR. HENRY BLUMBERG, University of Illinois.

FIRING DATA.

By J. K. WHITTEMORE.

In the first part of this paper, I shall outline a course in "Firing Data," and in the second part, make some suggestions as to the conduct of such a course. The outline is based on the course given in the last college year in the Yale R. O. T. C. The course in firing data includes all the mathematics necessary for an officer of the Field Artillery, and is an extremely important part of the training for a commission in that service. The work, as here described, applies to the U. S. 3-inch gun, the British 18 pounder, and the French 75 mm., but the tables used in the examples apply only to the U. S. 3″ gun; they are given in Danford and Moretti's "Notes on Training Field Artillery Details," Yale University Press, Sixth Printing, Feb., 1918.

The fundamental firing data problem is the computation of data for the battery for the opening salvo in indirect laying from observations made at the battery commander's station. These data are range, site, deflection and deflection difference. Indirect laying is pointing a gun at a target invisible at the gun; both target and battery must be visible at the battery commander's station.

The course begins with certain definitions which must be thoroughly learned

[1] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, 3. Auflage, Band I, Leipzig, 1907, pp. 227, 353. *Cf.* *L'Algèbre d'Omar Alkhayyâmî* publiée . . . par F. Woepeke, Paris, MDCCLI, pp. 94–96; A. Conti, in *Fragen der Elementargeometrie* (Enriques), Band 2, p. 215; and A. Adler, *l. c.*, 1906, p. 237.

[2] *La Géometrie*, livre II, *Oeuvres de Descartes* publiées par C. Adam et P. Tannery, Vol. 6, Paris, 1902, p. 391.

[3] E. Lill, (1) Résolution graphique des équations numériques de tous les degrés à une seule inconnue, et description d'un instrument inventé dans ce but," *Nouvelles annales de mathématiques*, 1867, (2), tome 6, pp. 359–362; (2) "Résolution graphique des équations numériques d'un degré quelconque à une inconnue," *Comptes rendus de l'académie des sciences*, Paris, tome 65, pp. 854–857. See also T. Vahlen, *Konstruktionen und Approximationen in systematischer Darstellung*, Leipzig, 1911, p. 141.